Fourier Series and Fejér's Theorem

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1 Introduction : Background and Motivation

A Fourier series can be understood as the decomposition of a periodic function into its projections onto an orthonormal basis. More precisely, consider the vector space of continuous functions from $[-\pi, \pi]$ to $\mathbb{R}$, on which we define the inner product between two functions f and g as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Then the Fourier series of a continuous, 2π-periodic function $f : [\pi, \pi] \to \mathbb{R}$ is

$$a_0 + \sum (a_n \cos(nt) + b_n \sin(nt))$$

where the coefficients (a_n) and (b_n) are given by

$$\begin{array}{rcl} a_0 = & \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx & \\ a_n = & \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx & \text{for } n > 0 \\ b_n = & \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx. & \end{array}$$

These coefficients are the projections of f onto the orthonormal basis functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos(mx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} \qquad m, n \in \mathbb{Z}.$$

Another way of expressing the the Fourier series of f is

$$\sum_n \hat{f}(n)e^{inx}$$

where the complex coefficients $\hat{f}(n)$ are given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

This complex exponential representation is equivalent to the trigonometric representation, and is a bit more compact. We can return to the trigonometric representation by taking the real and imaginary parts of this exponential representation. Also, by using DeMoivre's theorem, and defining $\hat{f}(n) = \frac{a_n - ib_n}{2}$ and $\hat{f}(-n) = \frac{a_n + ib_n}{2}$, we can change the trigonometric representation into the exponential one. In the following pages, we will use this exponential representation for reasons of simplicity.

Thus Fourier series allow us to represent a perhaps complicated periodic function as simply a linear combination of projections onto a basis. Such a compact representation has proven exceedingly useful in the analysis of many real-world systems involving periodic phenomena, such as waves propagating on a string, electrical circuits with oscillating current sources, and heat diffusion on a metal ring – an application we will later examine in detail. More generally, Fourier series usually arise in the ubiquitous context of boundary value problems, making them a fundamental tool among mathematicians, scientists, and engineers.

However, there is a caveat. Except in degenerate cases, a Fourier series is usually not an exact replica of its original function. Thus, a natural question is: exactly how does the series approximate the function? If we say that the Fourier series *converges* to the function, then precisely in what sense does the series converge? And under what conditions? Incidentally, such questions of Fourier series convergence are largely responsible for seeding the subject of real analysis.

One notion of convergence between functions is L^2-convergence, or convergence in the mean. For a 2π-periodic function f, we have L^2 convergence of the Fourier series if

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left|f(x) - \sum_{n=-N}^{N}\hat{f}(n)e^{inx}\right|^2 dx = 0.$$

One of the first results regarding Fourier series convergence is that if f is square-integrable (that is, if $\int_0^{2\pi}|f(x)|^2dx < \infty$), then its Fourier series L^2-converges to f. This is a nice result, but it leaves more to be desired. L^2 convergence only says that over the interval $[-\pi, \pi]$, the average deviation between f and its Fourier series must tend to zero. However, for a fixed x in $[-\pi, \pi]$, there are no guarantees on the difference between $f(x)$ and the series approximation at x.

A stronger – and quite natural – sense of convergence is *pointwise* convergence, in which we demand that at each point $x \in [-\pi, \pi]$, the series approximation converges to $f(x)$. Jordan's Pointwise Convergence Theorem then states that if f is sectionally continuous and x_0 is such that the one-sided derivatives $f'(x_0^+)$ and $f'(x_0^-)$ both exist, then the Fourier series $\sum_n \hat{f}(n)e^{inx_0}$ converges to $f(x_0)$. This theorem is often useful for proving pointwise convergence, and its conditions often hold. However, sometimes pointwise convergence can be an inappropriate notion of convergence. A canonical example is the sequence of functions defined by $g_n(x) : x \to x^n$ for $x \in [0, 1]$. Then (g_n) converges pointwise to a function h that equals 0 for $x \in [0, 1)$, but equals 1 for $x = 1$. Thus although (g_n) consists only of continuous functions, oddly the limit function is discontinuous.

To avoid such problems, we desire the even stronger notion of *uniform* convergence, such that the rate at which the series converges is identical for all points in $[-\pi, \pi]$. By adopting the metric

$$d(f, g) = \sup\{|f(x) - g(x)| : t \in [-\pi, \pi]\}$$

over the space of continuous functions from $[-\pi, \pi]$ to $\mathbb{R}$, we can force convergence to imply uniform convergence, simply by definition. This metric space is denoted by $C([-\pi, \pi], \mathbb{R})$. It can also be proven that $C([-\pi, \pi], \mathbb{R})$ is a vector space, and thus the concept of series is well-defined.

We are now primed to appreciate Fejér's remarkable theorem.

Fejér's Theorem: *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function with $f(-\pi) = f(\pi)$. Then the Fourier series of f (C,1)-converges to f in $C([-\pi, \pi], \mathbb{R})$, where $C([-\pi, \pi], \mathbb{R})$ is the metric space of continuous functions from $[-\pi, \pi]$ to $\mathbb{R}$.*

Without imposing any additional conditions on f aside from being continuous and periodic, Fejér's theorem shows that Fourier series can still achieve uniform convergence, granted that we instead consider the arithmetic means of partial Fourier sums.

2 Proof

2.1 Fejér's Kernel

Before proceeding further, we first prove some properties of Fejér's kernel – a trigonometric polynomial that often appears in Fourier analysis. These properties will be useful in the proof of Fejér's theorem.

Fejér's kernel can be expressed in either of the following two equivalent ways:

$$F_n(x) = \frac{1}{n+1} \frac{\sin^2[(n+1)x/2]}{\sin^2[x/2]} \tag{1}$$

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) \tag{2}$$

where $D_k(x)$ is the Dirichlet kernel $D_k(x) = \sum_{m=-k}^{k} e^{imx}$. Depending on the circumstances, one form of Fejér's kernel can lend more clarity than the other. Conversion between the two forms is just a tedious exercise in manipulating trigonometric identities. To avoid detracting from the flow of our presentation, we will not present the proof of this conversion here. However, the meticulous reader is welcome to read the proof in the **Appendix** section.

Lemma: The Fejér kernel has the following properties:

i $$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx = 1 \tag{3}$$

ii $$F_n(x) \geq 0 \tag{4}$$

iii For each fixed $\delta > 0$, $$\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} F_n(x) dx = 0 \tag{5}$$

Proofs:

i $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx = 1$

We appeal to the second form of Fejér's kernel given by (2). Substituting the definition of Dirichlet's kernel yields:

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{m=-k}^{k} e^{imx}.$$

Integrating $F_n(x)$ then yields

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{n+1} \sum_{k=0}^{n} \sum_{m=-k}^{k} e^{imx} \right] dx \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{m=-k}^{k} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} dx \right]. \end{aligned}$$

When m is nonzero, $\int_{-\pi}^{\pi} e^{imx} = 0$. But when $m = 0$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx} = 1$. Thus,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} 1 = 1. \quad \square$$

ii $F_n(x) \geq 0$

The non-negativity of the Fejér kernel follows immediately from the first form of the Fejér kernel (1). $\square$

iii For each fixed $\delta > 0, \lim_{n\to\infty}\int_{\delta\leq|x|\leq\pi} F_n(x)dx = 0$.

We again use the Fejér kernel's first form. For $\delta \leq |t| \leq \pi$ we have $\frac{1}{\sin^2 x/2} \leq \frac{1}{\sin^2 \delta/2}$. Thus

$$0 \leq F_n(x) \leq \frac{1}{n+1}\frac{1}{\sin^2 \delta/2}, \qquad \delta \leq |x| \leq \pi.$$

This uniformly converges to 0 as $n \to \infty$. $\square$

2.2 Fejér's Theorem

To discuss Cesaro convergence of Fourier series, we introduce notation for both the partial Fourier sums, and the arithmetic means of those partial sums. Denote the n^{th} partial sum of the Fourier series by s_n, and denote the corresponding n^{th} Cesaro sum by σ_n.

$$s_n(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx} \tag{6}$$

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(x) \tag{7}$$

Now we aim to rewrite these expressions in terms of the Fejér kernel. Substituting the integral form of $\hat{f}(k)$ into (6) yields

$$\begin{aligned} s_n(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx} &= \sum_{k=-n}^{n}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt\right]e^{ikx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{k=-n}^{n} e^{ik(x-t)}dt \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt.$$

Applying a change of variables then produces

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt. \tag{8}$$

With (8) in hand, we rewrite the Cesaro sum σ_n as

$$\begin{aligned}
\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(x) &= \frac{1}{n+1}\sum_{k=0}^{n}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_k(t)dt\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left[\frac{1}{n+1}\sum_{k=0}^{n} D_k(t)\right] dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)dt
\end{aligned}$$

By Property **i** of the Lemma, we can then write

$$\sigma_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t)-f(x))F_n(t)dt.$$

Applying the triangle inequality for integrals yields

$$|\sigma_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|(f(x-t)-f(x))F_n(t)\right| dt.$$

By the non-negativity of the Fejér kernel (Lemma **ii**), this reduces to

$$|\sigma_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x-t)-f(x)\right| F_n(t)dt.$$

Continuous functions on $[-\pi,\pi]$ are uniformly continuous. That is, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|x-y| \leq \delta$ implies $|f(x) - f(y)| \leq \epsilon$. We now break our integral into two integrals, with the limits of integration divided about δ and $-\delta$.

$$|\sigma_n(x) - f(x)| = \left(\frac{1}{2\pi}\int_{|t|\leq\delta}\left|f(x-t)-f(x)\right| F_n(t)dt\right) + \left(\frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi}\left|f(x-t)-f(x)\right| F_n(t)dt\right).$$

From the uniform continuity of f, the first integral is bounded above by

$$\frac{1}{2\pi}\int_{|t|\leq\delta}\epsilon F_n(t)dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\epsilon F_n(t)dt = \epsilon$$

where the last equality holds by Lemma **i**.

If we let $M = \sup_{-\pi\leq t\leq\pi}|f(t)|$, then the second integral is bounded above by

$$\frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} 2MF_n(t)dtF_n(t)dt = \frac{M}{\pi}\int_{\delta\leq|t|\leq\pi} F_n(t)dt.$$

Finally by Lemma **iii**, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\int_{\delta\leq|t|\leq\pi} F_n(t)dt \leq \epsilon$. Conclusively, for all $n \geq N$, $|f(x) - \sigma_n(x)| \leq \epsilon + \epsilon = 2\epsilon$. This completes the proof. $\square$

3 Application: Heat Diffusion on a Circle

In this section, we examine one of the very first applications of Fourier series. It dates from Fourier's seminal 1807 paper "The Analytical Theory of Heat", in which Fourier series are used to solve the practical problem of heat flow in various metallic solids.

Imagine a wire of unit length that is twisted into a circle. Suppose this circle is heated by some continuous initial temperature distribution f. As time passes, the heat redistributes itself about the circle, moving from hotter areas to colder areas. After a long time, we would expect the heat to be evenly distributed over the circle. But in the interim, we would like an expression for the temperature as a function of both space and time.

It is convenient to think of this circle as the unit interval $[0, 1]$ wrapped around on itself. We then denote the temperature function by $u(x, t)$, where x is the spatial variable lying on the real line modulo 1, and t is time. The initial condition is then $u(x, 0) = f(x)$. Note that the circularity forces $u(x, t)$ to be periodic in x with period 1, for any fixed t. Thus $u(x, t)$ has some Fourier series expansion

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t)e^{2\pi inx}$$

where the Fourier coefficients are given by

$$c_n(t) = \int_0^1 e^{-2\pi inx}u(x,t)dx.$$

At this point we recall Newton's famous heat conduction equation, which approximates the conduction of heat in solids. The equation is $\alpha^2 u_{xx} = u_t$, where α^2 is the thermal diffusivity constant; to simplify matters, we will let $\alpha^2 = \frac{1}{2}$. With the intent of applying this equation, we first differentiate $c_n(t)$ with respect to t

$$c_n'(t) = \int_0^1 u_t(x,t)e^{-2\pi inx}dx$$

and then substitute the heat conduction equation

$$c_n'(t) = \int_0^1 \frac{1}{2}u_{xx}(x,t)e^{-2\pi inx}dx.$$

Now we would like to remove the spatial derivatives from u. To do this we integrate by parts twice, using the facts that $e^{2\pi in} = 1$, and $u(0,t) = u(1,t)$ by periodicity of u. After integrating by parts we have

$$\begin{aligned} c_n'(t) &= \int_0^1 \frac{1}{2}u(x,t)\frac{d^2}{dx^2}\left[e^{-2\pi inx}\right]dx \\ &= \int_0^1 \frac{1}{2}u(x,t)(-4\pi^2n^2)e^{-2\pi inx}dx \\ &= (-2\pi^2n^2)\int_0^1 u(x,t)e^{-2\pi inx}dx \\ &= (-2\pi^2n^2)c_n(t) \end{aligned}$$

To our approval, we discover that $c_n(t)$ obeys a canonical ordinary differential equation! Its solution is of course

$$c_n(t) = c_n(0)e^{-2\pi^2n^2t}.$$

Expressing $c_n(0)$ in integral form shows that $c_n(0)$ is simply the n^{th} Fourier coefficient of the initial distribution function f:

$$c_n(0) = \int_0^1 u(x,0)e^{-2\pi inx}dx = \int_0^1 f(x)e^{-2\pi inx}dx.$$

Denoting this coefficient by $\hat{f}(n)$, we can elegantly write the general solution of the heat equation as

$$u(x,t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-2\pi^2 n^2 t} e^{2\pi i n x}.$$

4 Appendix: Equivalent Forms of Fejér's Kernel

Recall that Fejér's kernel can be expressed as either

$$F_n(x) = \frac{1}{n+1} \frac{\sin^2[(n+1)x/2]}{\sin^2[x/2]}$$

or

$$F_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x)$$

where $D_k(x)$ is the Dirichlet kernel $D_k(x) = \sum_{m=-k}^{k} e^{imx}$. In this section we prove the equivalence of these expressions. Namely, we will manipulate the second form of Fejér's kernel listed above into the first form.

We start with the following Lemma:

Lemma:

$$1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin[(n+\frac{1}{2})x)]}{\sin(\frac{x}{2})}.$$

Proof: Recall the trigonometric product identity $2\cos(u)\sin(v) = \sin(u+v) - \sin(u-v)$. Setting $u = kx$ and $v = \frac{x}{2}$, we then have

$$2\cos(kx) = \frac{\sin\left[\left(k+\frac{1}{2}\right)x\right] - \sin\left(\left(k-\frac{1}{2}\right)x\right)}{\sin\left[\frac{x}{2}\right]}$$

By substituting the above expression for $2\cos(kx)$, we then have a telescoping sum

$$\begin{aligned} 2\sum_{k=1}^{n} \cos(kx) &= \sum_{k=1}^{n} \frac{\sin\left[\left(k+\frac{1}{2}\right)x\right] - \sin\left[\left(k-\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)} \\ &= \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right] - \sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \end{aligned}$$

$$= \frac{\sin[(n+\frac{1}{2})x)]}{\sin(\frac{x}{2})} - 1$$

which yields the result. $\square$

Using this Lemma and De Moivre's formula, we can now rewrite Dirichlet's kernel as

$$D_k(x) = \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin[(n+\frac{1}{2})x)]}{\sin(\frac{x}{2})}.$$

Substituting this into the second form of Fejér's kernel yields

$$\begin{aligned}
(n+1)F_n(x) &= \sum_{k=0}^{n} D_k(x) \\
&= \sum_{k=0}^{n} \frac{\sin[(n+\frac{1}{2})x)]}{\sin(\frac{x}{2})} \\
&= \frac{1}{\sin(x/2)} \mathrm{Im}\left\{\sum_{k=0}^{n} e^{i(k+1/2)x}\right\} \\
&= \frac{1}{\sin(x/2)} \mathrm{Im}\left\{e^{ix/2}\frac{e^{i(n+1)x}-1}{e^{ix}-1}\right\} \\
&= \frac{1}{\sin(x/2)} \mathrm{Im}\left\{\frac{e^{i(n+1)x}-1}{e^{ix/2}-e^{-ix/2}}\right\} \\
&= \frac{1-\cos[(n+1)x]}{2\sin^2(x/2)} \\
&= \frac{\sin^2[(n+1)x/2]}{\sin^2[x/2]}. \quad \square
\end{aligned}$$

References

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[3] Brad E. Osgood. *Lecture notes for EE261,* (Stanford University, 2003).