PROJECTILE MOTION ON A CART
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Problem A cart is moving on an frictionless incline sloped with angle ϕ, with initial velocity v_0 in the direction of the incline. A projectile is launched in the direction normal to the incline's surface, with velocity v. Prove that the projectile lands on the cart.

Solution Let θ denote the angle of the projectile's initial velocity with respect to the horizontal, as shown in Figure 1.

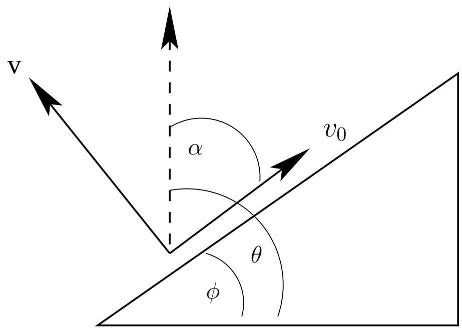

Figure 1. Projectile Motion on Moving Cart

Using results from a previous problem, we know that the length up the incline that the projectile will travel is

$$L = \frac{2(v_0^2 + v^2)\cos^2\theta(\tan\theta - \tan\phi)}{g\cos\phi} \tag{1}$$

where $(v_0^2 + v^2)$ is the squared magnitude of the projectile's initial velocity. We now compute the amount time required for the projectile to land at this location. Since the only force acting on the system is gravitational acceleration in the vertical direction, the velocity of the projectile in the horizontal direction is constant, and is given by

$$v_x = \sqrt{v_0^2 + v^2}\cos\theta.$$

Since the distance the projectile travels up the incline is L, the horizontal distance that the projectile travels is $L\cos\phi$. The total flight time of the projectile is then

$$\tau = \frac{L\cos\phi}{v_x} = \frac{L\cos\phi}{\sqrt{v_0^2 + v^2}\cos\theta}. \tag{2}$$

In the meantime, the distance up the incline that the cart travels is

$$S = -\frac{1}{2}(g\sin\phi)\tau^2 + v_0\tau. \tag{3}$$

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We wish to prove that $S = L$. Assuming that this is true, substituting our equation for S and τ yields

$$-\frac{1}{2}(g\sin\phi)\tau^2 + v_0\tau = L$$

$$-\frac{1}{2}(g\sin\phi)\left(\frac{L\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right)^2 + v_0\left(\frac{L\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right) = L$$

$$-\frac{1}{2}(g\sin\phi)\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right)^2 L + v_0\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right) = 1$$

where in the last step we cancelled an L from both sides. Substituting Equation 1 yields

$$-\frac{1}{2}(g\sin\phi)\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right)^2\left(\frac{2(v_0^2+v^2)\cos^2\theta(\tan\theta-\tan\phi)}{g\cos\phi}\right) + v_0\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right) = 1. \tag{4}$$

It thus suffices to argue that the left-hand side (LHS) is indeed equal to 1. Cancelling terms in the numerators and denominators, we have:

$$\begin{aligned}
LHS &= -\frac{1}{2}(g\sin\phi)\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right)^2\left(\frac{2(v_0^2+v^2)\cos^2\theta(\tan\theta-\tan\phi)}{g\cos\phi}\right) + v_0\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right) \\
&= -(g\sin\phi)\left(\frac{\cos\phi}{\cos\theta}\right)^2\left(\frac{\cos^2\theta(\tan\theta-\tan\phi)}{g\cos\phi}\right) + v_0\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right) \\
&= -(g\sin\phi)\left(\frac{\cos\phi(\tan\theta-\tan\phi)}{g}\right) + v_0\left(\frac{\cos\phi}{\sqrt{v_0^2+v^2}\cos\theta}\right) \\
&= -\sin\phi\cos\phi(\tan\theta-\tan\phi) + \frac{v_0}{\sqrt{v_0^2+v^2}}\left(\frac{\cos\phi}{\cos\theta}\right)
\end{aligned}$$

Observe that $\cos\alpha = \frac{v_0}{\sqrt{v_0^2+v^2}}$. Thus,

$$\begin{aligned}
LHS &= -\sin\phi\cos\phi(\tan\theta-\tan\phi) + \left(\frac{\cos\alpha\cos\phi}{\cos\theta}\right) \\
&= -\sin\phi\cos\phi(\tan\theta-\tan\phi) + \left(\frac{\cos(\theta-\phi)\cos\phi}{\cos\theta}\right) \\
&= -\sin\phi\cos\phi(\tan\theta-\tan\phi) + \left(\frac{(\cos\theta\cos\phi+\sin\theta\sin\phi)\cos\phi}{\cos\theta}\right) \\
&= -\sin\phi\cos\phi(\tan\theta-\tan\phi) + (\cos\phi+\tan\theta\sin\phi)\cos\phi \\
&= -\sin\phi\cos\phi\tan\theta + \sin\phi\cos\phi\tan\phi + \cos^2\phi + \sin\phi\cos\phi\tan\theta \\
&= \sin\phi\cos\phi\tan\phi + \cos^2\phi \\
&= \sin^2\phi + \cos^2\phi \\
&= 1.
\end{aligned}$$

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