# Falling Factorials, Generating Functions, and Conjoint Ranking Tables

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# Outline of Talk

# Outline

1. Fourier Motivations
   - Big Picture
   - Time-Limited and Bandlimited

2. Combinatorics of Falling Factorials
   - Results
   - Open Problems

# Motivations from Discrete Fourier Analysis

## Ultimate Goal

Discrete analogs for all major theorems in classical Fourier analysis.

| concept | continuous | discrete |
|---|---|---|
| fourier transform | $X(f) := \int x(t)e^{i2\pi ft}dt$ | $X[m] := \sum_{n=0}^{N} x[n]\exp(i\frac{2\pi mn}{N})$ |
| convolution | $\int x(\lambda)y(t-\lambda)d\lambda$ | circular convolution |
| delta function | dirac delta | kronecker delta |
| sinc | $\frac{\sin t}{t}$ | $\frac{1}{\lvert\mathcal{J}\rvert}\frac{\sin(\frac{\pi m\lvert\mathcal{J}\rvert}{N})}{\sin(\frac{\pi m}{N})}$ |
| fourier decay rate | $f \in C^p \leftrightarrow \lvert\mathcal{F}f(s)\rvert = O(\frac{1}{\lvert s\rvert^p})$ | ??? |
| sampling theorem | $x(t) = \sum_{n\in\mathbf{Z}} x[n]sinc(t-n)$ | ??? (my thesis topic) |
| uncertainty principles | signal cannot be both time-limited and bandlimited | $N_t N_w \geq N$ (donoho, 1989) ??? |

# Towards Discrete Uncertainty

"A continuous signal cannot be time-limited and bandlimited."

## Theorem

*Suppose $f$ is a continuous bandlimited signal.*
*If $f(t) = 0$ for all $t \in (a, b)$, then $f \equiv 0$.*

## Proof

Let $\mathcal{F}f(s) = 0$ for $|s| \geq p/2$, and let $t \in (a, b)$. Then

$$0 = f(t) = \int_{-\infty}^{\infty} \mathcal{F}f(s) e^{2\pi i s t} ds = \int_{-p/2}^{p/2} \mathcal{F}f(s) e^{2\pi i s t} ds.$$

Repeatedly differentiate with respect to $t$. Then

$$0 = (2\pi i)^n \int_{-p/2}^{p/2} \mathcal{F}f(s) s^n e^{2\pi i s t} ds.$$

Pick $t_0 \in (a, b)$. Lemma: $\int_{-p/2}^{p/2} \mathcal{F}f(s) s^n e^{2\pi i s t_0} ds = 0$. (Put this aside for later.)

## Proof (continued)

Now for *any* $t$, we can write:

$$\begin{aligned} f(t) &= \int_{-p/2}^{p/2} \mathcal{F}f(s) e^{2\pi i s t} ds \\ &= \int_{-p/2}^{p/2} \mathcal{F}f(s) e^{2\pi i s (t-t_0)} e^{2\pi i s t_0} ds \\ &= \int_{-p/2}^{p/2} \sum_{n=0}^{\infty} \frac{(2\pi i s (t-t_0))^n}{n!} e^{2\pi i s t_0} \mathcal{F}f(s) ds \\ &= \sum_{n=0}^{\infty} \frac{(2\pi i (t-t_0))^n}{n!} \underbrace{\int_{-p/2}^{p/2} s^n e^{2\pi i s t_0} \mathcal{F}f(s) ds}_{Lemma} \\ &= \sum_{n=0}^{\infty} \frac{(2\pi i (t-t_0))^n}{n!} \cdot 0 = 0. \quad \square \end{aligned}$$

- Key Moves:
  1. Fourier inversion formula
  2. Repeated differentiation at $t_0$ where $f(t_0) = 0$
  3. Taylor Expansion for $e^{2\pi i s(t-t_0)}$
  4. Peel $s^n$ out of $(2\pi i s(t-t_0))^n$
  5. Interchange sum and integral (uniform convergence)
- **Open Problem**: Replicate this chess game in discrete world.
- Replace derivatives with differences, and Taylor with Newton:

$$g[m] = \sum_{n=0}^{\infty} \frac{(\Delta^n g)[0]}{n!} m^{\underline{n}}.$$

For arbitrary $m_1$, we have

$$\begin{aligned} f[m_1] &= \frac{1}{N}\sum_{n=0}^{N-1} F[n]\cdot \underline{\omega}[m_1 n] \\ &= \frac{1}{N}\sum_{n=0}^{N-1} F[n]\cdot \underline{\omega}[m_0 n]\cdot \underline{\omega}[(m_1-m_0)n] \\ &= \frac{1}{N}\sum_{n=-p}^{p} F[n]\cdot \underline{\omega}[m_0 n]\cdot \left(\sum_{k=0}^{(m_1-m_0)n} \frac{\eta^k}{k!}((m_1-m_0)n)^{\underline{k}}\right) \end{aligned}$$

where $\eta := e^{2\pi i/N} - 1$.
Need to peel $n^{\underline{k}}$ out of the falling factorial!

## Problem

*How do we write $(xy)^{\underline{k}}$ in terms of $x^{\underline{l}}$ and $y^{\underline{m}}$?*

# Outline

# Problem

- Let $x^{\underline{k}}$ denote the falling factorial,

$$x^{\underline{k}} = x(x-1)\dots(x-k+1)\,, \quad k \in \mathbb{N}_+$$

- In the equation

$$(xy)^{\underline{k}} = \sum_{l,m=1}^{k} c_{l,m}^{(k)} x^{\underline{l}} y^{\underline{m}}$$

what can we say about the coefficients $c_{l,m}^{(k)}$?

- $$\begin{aligned}(xy)^{\underline{2}} &= (xy)(xy-1)\\ &= 1x^{\underline{1}}y^{\underline{2}} + 1x^{\underline{2}}y^{\underline{1}} + 1x^{\underline{2}}y^{\underline{2}}.\\ (xy)^{\underline{3}} &= (xy)(xy-1)(xy-2)\\ &= 1x^{\underline{1}}y^{\underline{3}} + 1x^{\underline{3}}y^{\underline{1}} + 6x^{\underline{2}}y^{\underline{2}} + 3x^{\underline{2}}y^{\underline{3}} + 3x^{\underline{3}}y^{\underline{2}} + 1x^{\underline{3}}y^{\underline{3}}.\end{aligned}$$

$$
\begin{aligned}
(xy)^{\underline{9}} &= xy(xy-1)(xy-2)\cdots(xy-8) \\
&= 1x^{\underline{9}}y^{\underline{1}} + 1x^{\underline{1}}y^{\underline{9}} + 255x^{\underline{9}}y^{\underline{2}} + 255x^{\underline{2}}y^{\underline{9}} \\
&\quad + 4608x^{\underline{8}}y^{\underline{2}} + 4608x^{\underline{2}}y^{\underline{8}} + 3025x^{\underline{9}}y^{\underline{3}} + 3025x^{\underline{3}}y^{\underline{9}} \\
&\quad + 24192x^{\underline{7}}y^{\underline{2}} + 24192x^{\underline{2}}y^{\underline{7}} + 74124x^{\underline{8}}y^{\underline{3}} + 74124x^{\underline{3}}y^{\underline{8}} + \ldots
\end{aligned}
$$

Coefficients in expansion of $(xy)^{\underline{9}}$, displayed as a symmetric matrix:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 15120 & 40320 & \boxed{24192} & \boxed{4608} & \boxed{255} \\
0 & 0 & 10080 & 544320 & 1958040 & 1796760 & 588168 & \boxed{74124} & \boxed{3025} \\
0 & 0 & 544320 & 6108480 & 12267360 & 7988904 & 2066232 & 218484 & 7770 \\
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0 & \boxed{4608} & \boxed{74124} & 218484 & 212436 & 85680 & 15624 & 1260 & 36 \\
\boxed{1} & \boxed{255} & \boxed{3025} & 7770 & 6951 & 2646 & 462 & 36 & 1
\end{pmatrix}
$$

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\end{pmatrix}$$

Observations:

- all nonnegative
- hyperbolic phase transition
- unimodality and log-concavity

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Observations:

- all nonnegative
- hyperbolic phase transition
- unimodality and log-concavity
- Stirling Numbers of the 2nd Kind in last row

**WHY???**

# The Combinatorial Meaning: Quals!

- Quals preference form = table of $l$ professors and $m$ subjects.
- You must *rank* your top $k$ choices. However, you are not allowed to completely avoid any professors or subjects.
- Example: $l = 4$, $m = 5$, and $k = 9$

| | signals | systems | circuits | physics | cs |
|---|---|---|---|---|---|
| Thomas Cover | 3 | 6 | | 8 | |
| John Gill | 2 | 5 | | | 7 |
| Brad Osgood | 1 | 4 | | | |
| Charles Xavier | | | 9 | | |

- In marketing, this is called a "conjoint ranking table."

## Theorem (Combinatorial Characterization)

*$l!m! \cdot c_{l,m}^{(k)}$ = number of ways to make $k$ ranked choices in an $l \times m$ quals form, without avoiding any subjects or professors.*

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| | signals | systems | circuits | physics | cs |
|---|---|---|---|---|---|
| Thomas Cover | 3 | 6 | | 8 | |
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| Charles Xavier | | | 9 | | |

- # of possible forms $= 4! \cdot 5! \cdot \underbrace{c_{4,5}^{(9)}}_{12267360} = 35,329,996,800.$
- Compare with $\binom{20}{9} 9! = 60,949,324,800.$
- $(xy)^{\underline{k}}$ is a **Generating Function** for quals preference forms!

# Phase Transition

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 15120 & 40320 & 24192 & 4608 & 255 \\ 0 & 0 & 10080 & 544320 & 1958040 & 1796760 & 588168 & 74124 & 3025 \\ 0 & 0 & 544320 & 6108480 & 12267360 & 7988904 & 2066232 & 218484 & 7770 \\ 0 & 15120 & 1958040 & 12267360 & 18329850 & 9874746 & 2229402 & 212436 & 6951 \\ 0 & 40320 & 1796760 & 7988904 & 9874746 & 4690350 & 965790 & 85680 & 2646 \\ 0 & 24192 & 588168 & 2066232 & 2229402 & 965790 & 185766 & 15624 & 462 \\ 0 & 4608 & 74124 & 218484 & 212436 & 85680 & 15624 & 1260 & 36 \\ 1 & 255 & 3025 & 7770 & 6951 & 2646 & 462 & 36 & 1 \end{pmatrix}$$

**Corollary (Hyperbolic Phase Transition)**

$$c_{l,m}^{(k)} = 0 \quad \textit{for} \quad lm < k \quad \textit{and} \quad c_{l,m}^{(k)} > 0 \quad \textit{for} \quad lm \geq k\,.$$

**Proof.**

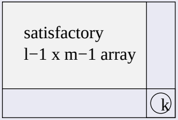
If $l \cdot m < k$, there aren't enough cells to fill in $k$ choices. Done! □

# Recurrence Relation

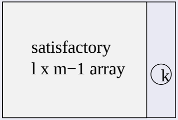
## Theorem (Recurrence Relation)

$$c_{l,m}^{(k)} = c_{l-1,m-1}^{(k-1)} + l c_{l,m-1}^{(k-1)} + m c_{l-1,m}^{(k-1)} + (lm - (k-1)) c_{l,m}^{(k-1)} .$$

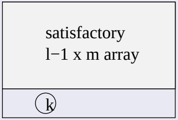
## Proof.



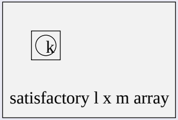

Case (1):
Ball "k" is alone in both its row and its column




Case (3):
Ball "k" is alone in its column but not its row




Case (2):
Ball "k" is alone in its row but not its column




Case (4):
Ball "k" is neither alone in its row nor its column

□

# Explicit Formulae

$$c_{l,m}^{(k)} = \sum_{p=1}^{k} (-1)^{k-p} \begin{bmatrix} k \\ p \end{bmatrix} \begin{Bmatrix} p \\ l \end{Bmatrix} \begin{Bmatrix} p \\ m \end{Bmatrix} \tag{1}$$

$$c_{l,m}^{(k)} = \frac{k!}{l!m!} \sum_{h=0}^{l+m} \sum_{p=0}^{h} (-1)^h \binom{l}{p} \binom{m}{h-p} \binom{(l-p)(m-h+p)}{k} \tag{2}$$

$$c_{l,m}^{(k)} = \frac{k!}{l!m!} \sum_{s,t=1}^{k} (-1)^{l+s+m+t} \binom{l}{s} \binom{m}{t} \binom{st}{k} \tag{3}$$

1. Stirling Numbers and Change of Basis
2. Inclusion-Exclusion Principle
3. Mobius Inversion and Matrix Inversion

# Open Problems

## Open Problem I

Armed with these relations for the coefficients in

$$(xy)^{\underline{k}} = \sum_{1 \leq l, m \leq k} c_{l,m}^{(k)} x^{\underline{l}} y^{\underline{m}},$$

find the discrete analogue to Dym and McKean's proof that a signal cannot be both time-limited and bandlimited.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 15120 & 40320 & 24192 & \boxed{4608} & 255 \\
0 & 0 & \boxed{10080} & 544320 & 1958040 & 1796760 & \boxed{588168} & 74124 & 3025 \\
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\end{pmatrix}$$

**Open Problem II (Striking Conjecture)**

Rows, columns, diagonals, and antidiagonals are all log-concave.

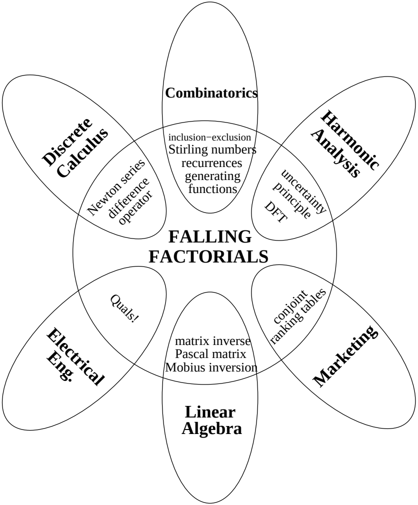
- Definition: $(a_i)$ is log-concave iff $a_{i+1}a_{i-1} \leq a_i^2$.
- Hint: Negative Root Test.

# Conclusions

- "A signal cannot be both time-limited and bandlimited"
  $\Longrightarrow$ Discrete Analogue?
  $\Longrightarrow$ Motivation to study:

$$(xy)^{\underline{k}} = \sum_{1 \le l, m \le k} c_{l,m}^{(k)} x^{\underline{l}} y^{\underline{m}}$$

- Combinatorial Characterization
- Phase Transition
- Recurrence Relation
- Explicit Formulas
- Open Problems




# Thanks for listening!

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*Comments or Questions?*

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