Real Analysis Crash Course
Lecture 1: The Reals

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0 Introduction

First, here is an outline of topics I plan to cover in these sessions, so you know where we're headed.

1. Definition and Properties of $\mathbb{R}$
2. Sequences and Series
3. Metric Spaces
4. Topology Basics
5. Compactness
6. Connectedness
7. Uniform Convergence
8. † Inverse and Implicit Function Theorem
9. † Fubini's Theorem and Change of Variables
10. † Lebesgue Theory

The topics marked with † are not even reached in a quarter-long course. However, I'm hoping to extract just the most significant bits out of these topics, so hopefully we'll learn about some of the later topics.

1 Definition and Properties of $\mathbb{R}$

This lecture will be about defining the real number system ($\mathbb{R}$). If someone asked you to define $\mathbb{R}$, what would you say? Well, you might simply say that it is the field of all rational and irrational numbers. But today we will give a definition of $\mathbb{R}$ that is well suited for real analysis. Namely, the reals are the only ordered field satisfying certain completeness properties.

Why go through the trouble of defining $\mathbb{R}$ this way? Well, real analysis is in some sense all about ϵ, and what happens at limits. The motivation for defining $\mathbb{R}$ is to construct an ordered field that is rich enough for us to be able perform analysis, and take limits without running into serious badness. For instance, if we were to work strictly over the rationals instead of the reals, we could concoct all kinds of sequences whose limits don't even exist in the field. For instance, the sequence of rational approximations to π does not converge to a rational value. (In fact it "converges" to π.) Thus we want a system of numbers that is **complete**, where completeness intuitively means that there are no such gaps – that the limits of convergent sequences remain inside the field.

1.1 Monotone Sequence Property

Now say we want to precisely formulate what we mean by completeness. Simply saying that the limits of convergent sequences remain in the field is not good enough, because the statement itself is pathological. For instance, what would it mean to speak of the distance between an element in our field and an element outside the field? Such a distance may have no natural definition. Luckily, here is a careful phrasing that may seem round-about, but can express exactly what we want while avoiding such problems.

Monotone Sequence Property (MSP): Every monotonically increasing sequence which is bounded above converges.

Intuition: The sequence of rational approximations to π is an example of a monotonically increasing sequence which is bounded above. It does not converge to a rational, and thus the system of rationals fails this property.

1.1.1 Example: Proving Bolzano-Weierstrass

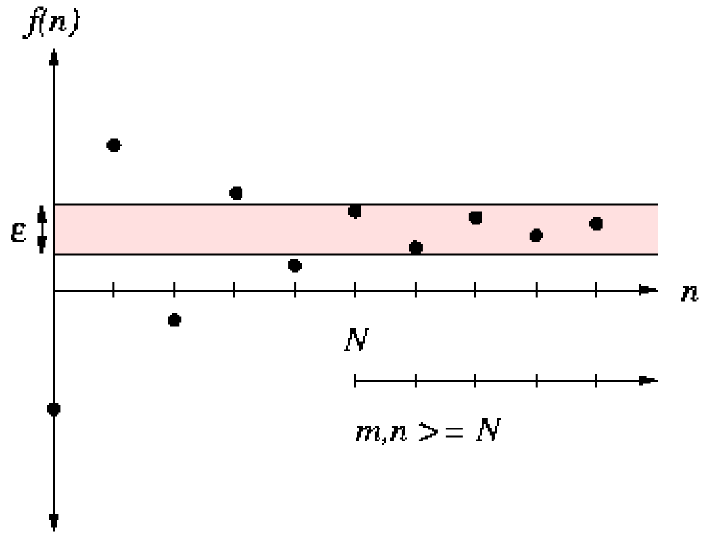
Sometimes the MSP is very useful in proofs. In fact, we will use this now to prove the following very important theorem:

Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a monotonic subsequence.

Proof: Super duper construction using repeated bisection of the bounded interval. Might make an image illustrating this later. Point out where we use the MSP!

1.2 Cauchy Criterion

Another notion of completeness is the Cauchy criterion. To describe it, first we must define a Cauchy sequence.



Cauchy Sequence: A sequence (x_n) is Cauchy if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n, m > N$ implies $|x_n - x_m| < \epsilon$. (A Cauchy sequence is sometimes called Cauchy-convergent.)

This definition looks similar to that of sequence convergence, but is quite different. In the definition of sequence convergence, we must show that the sequence gets arbitrarily close to some limit L. Here, we do not even have to know the limit to show Cauchy convergence. We just compare successive terms in the sequence. Pictorially, this is saying that if you ignore the first N terms of the Cauchy sequence, all remaining terms are bounded in an ϵ-width tunnel.

Now it should be clear from the definition that any convergent sequence is Cauchy. To show this, use an $\epsilon/2$ argument. From the definition of sequence convergence, we know there exists an N such that $\forall n > N$, $|x_n - L| < \epsilon/2$. Then by the triangle inequality, it follows that for all $j, k > N$, $|x_j - x_k| \leq |x_j - L| + |x_k - L| = \epsilon$. Thus the sequence is Cauchy.

However, one might also ask, it it possible that any Cauchy sequence is convergent? Looking at the illustrative picture, it seems that this might be true. We will call this the Cauchy Completeness Property:

Cauchy Completeness: If (x_n) is a Cauchy sequence, then it is convergent.

It turns out that $\mathbb{R}$ satisfies this property as well. In fact, one can show that the MSP implies Cauchy Completeness, and vice versa. Thus these notions of completeness are actually equivalent! Let us prove one half of the equivalence. Here is how to show that MSP $\Rightarrow$ Cauchy Completeness.

1. Every Cauchy sequence is bounded.
2. Every Cauchy sequence has a monotonic subsequence. (by Bolzano-Weierstrass, which is a consequence of the MSP)
3. By MSP, the previous two facts imply that this subsequence is convergent.
4. If a subsequence of a Cauchy sequence is convergent, then the original sequence converges as well to the same limit. Thus every Cauchy sequence is convergent.

As an exercise, prove the other direction: Cauchy Completeness $\Longrightarrow MSP$.

Conclusively, the take-home message is this. If someone asked you for a concise definition of the real number system, you can say that $\mathbb{R}$ **is the unique complete ordered field.** And completeness is understood to mean either that **bounded monotonic sequences are convergent**, or equivalently, that **all Cauchy sequences are convergent.**

These two completeness properties are very useful to prove convergence of a sequence. To use the first property, just show that the sequence is bounded and monotonic. To use the second, just compare successive terms in the sequence. In both cases, you do not need to find the limit, which could be a much more difficult task. Here is an example ...

1.2.1 Example: Proving Convergence of a Sequence

Let (x_n) be a sequence of reals such that $|x_n - x_{n+1}| \leq 1/2^n$. Show that x_n converges.

That concludes our discussion of $\mathbb{R}$.

2 Small Exercises

1. The Monotone Sequence Property can be equivalently stated as the

   Least Upper Bound Property: If S is a non-empty subset of $\mathbb{R}$ and is bounded above, then in $\mathbb{R}$ there exists a least upper bound for S.

   Show that these two properties are indeed equivalent.

2. Show that Cauchy Completeness implies the MSP.

3 Lecture Summary

- $\mathbb{R}$ is the unique complete ordered field.
- In $\mathbb{R}$, any monotonically increasing sequence that is bounded above is convergent.
- In $\mathbb{R}$, any bounded sequence has a convergent subsequence.
- In $\mathbb{R}$, any Cauchy sequence is convergent.
- In $\mathbb{R}$, any non-empty subset that is bounded above has a least upper bound.